Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

which is smallest in absolute value, is given by the series

$$-\frac{a_0}{a_1}-\frac{a_0^2a_2}{a_1\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}}-\frac{a_0^3\begin{vmatrix} a_2 & a_3 \\ a_1 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}\begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}}-\frac{a_0^4\begin{vmatrix} a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 & a_2 \\ 0 & 0 & a_0 & a_1 \end{vmatrix}}-\cdots.$$

In case of any algebraic equation with imaginary or complex roots the above formula clearly fails. State the conditions under which the formula may be relied upon to give correct results.

2804. Proposed by T. H. GRONWALL, Washington, D. C.

Show that for $|x| < 1$

$$\frac{1}{\sqrt{1-x^4}}\int_0^x\frac{dx}{\sqrt{1-x^4}} = x+\sum_1^\infty\frac{3\cdot7\cdots(4n-5)(4n-1)}{5\cdot9\cdots(4n-3)(4n+1)}x^{4n+1},$$

$$\left(\int_0^x\frac{dx}{\sqrt{1-x^4}}\right)^2 = x^2+\sum_1^\infty\frac{3\cdot7\cdots(4n-5)(4n-1)}{5\cdot9\cdots(4n-3)(4n+1)}\cdot\frac{x^{4n+2}}{2n+1}.$$

2805. Proposed by C. N. MILLS, Brookings, S. Dakota.

Derive the expression for volume

$$v = \int\int\int\rho^2\sin\phi d\rho d\phi d\theta.$$

In Byerly's *Integral Calculus*, page 183, revised edition, is a method by revolution, and in Czubers's *Integralrechnung*, page 200, is a method using the Jacobian determinant.

Required, a simple method one might use in developing the volume integral in polar coordinates.

2806. Proposed by R. E. MORITZ, University of Washington.

An anthropologist told me recently that large numbers of Russian peasants, whose knowledge of numbers is limited to multiplication and division by 2, employ the following method of multiplication which they were taught by a priest.

(1) Write the two numbers to be multiplied in the same horizontal line.

(2) Multiply the first number by 2 and write the product under the number so multiplied.

(3) Divide the second number by 2, discarding the remainder 1 when it occurs, and write the quotient under the number so divided.

(4) Treat the product and quotient thus obtained in the same manner as the original numbers. Continue this process until the quotient 1 is obtained.

(5) Strike out all the numbers on the left for which the corresponding numbers on the right are even.

(6) Add the remaining numbers on the left. Their sum is the required product.

Problem: Prove that this rule is correct.

SOLUTIONS OF PROBLEMS.

442. (Geometry) [1914, 156; 1919, 268]. Proposed by J. B. SMITH, Hampden-Sidney College.

If any three straight lines *AD*, *BE*, *CF*, be drawn from the corners of the triangle *ABC* to the opposite sides *a*, *b*, *c*, they will enclose an area. If Δ, Δ'' be the areas of the triangles *ABC*, *DEF*, show that

$$\frac{\Delta''}{\Delta}=\frac{(AF\cdot BD\cdot CE-AE\cdot CD\cdot BF)^2}{(ab-CE\cdot CD)(bc-AE\cdot AF)(ca-BF\cdot BD)},$$

where the signs of the factors are to be determined by the following rule: Each segment being measured from one of the corners of the triangle *ABC*, along one of the sides, is to be regarded as positive or negative according as it is drawn towards or from the other corner in that side.

I. Solution by Frank Irwin, University of California.

The proposition is incorrectly stated: D, E, F being the points of intersection with the opposite sides of the lines drawn through the vertices, Δ'' should be the area of the triangle, $A'B'C'$, formed by these three lines (not of DEF).

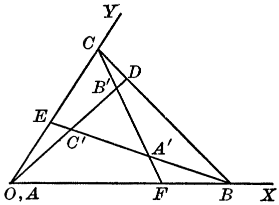
Taking AB, AC as the axes of x, y, $y = mx$ as the equation of AD, and $AE = b_2$, $AF = c_1$, the problem is a straightforward piece of analytic geometry. We readily find that D is the point $[bc/(b + cm), mbc/(b + cm)]$; A', $[(b - b_2)cc_1/(bc - b_2c_1), (c - c_1)bb_2/(bc - b_2c_1)]$; B', $[bc_1/(b + c_1m), mbc_1/(b + c_1m)]$; C', $[b_2c/(b_2 + cm), mb_2c/(b_2 + cm)]$.

The area of a triangle with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) being given by the expression

$$\tfrac{1}{2}\sin A \cdot \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix},$$

we have

$$\frac{\Delta''}{\Delta} = \frac{1}{bc} \cdot \begin{vmatrix} \dfrac{(b - b_2)cc_1}{bc - b_2c_1} & \dfrac{(c - c_1)bb_2}{bc - b_2c_1} & 1 \\ \dfrac{bc_1}{b + c_1m} & \dfrac{mbc_1}{b + c_1m} & 1 \\ \dfrac{b_2c}{b_2 + cm} & \dfrac{mb_2c}{b_2 + cm} & 1 \end{vmatrix}.$$



If we subtract m times the first column from the second, we get

$$\frac{\Delta''}{\Delta} = \frac{1}{bc} \cdot \frac{(c - c_1)bb_2 - m(b - b_2)cc_1}{bc - b_2c_1} \cdot \left(\frac{b_2c}{b_2 + cm} - \frac{bc_1}{b + c_1m}\right) = \frac{[(c - c_1)bb_2 - m(b - b_2)cc_1]^2}{bc(bc - b_2c_1)(b_2 + cm)(b + c_1m)}.$$

On the other hand we find, by similar triangles, that $BD = acm/(b + cm)$; whence $CD = ab/(b + cm)$. Also $BF = c - c_1$, $CE = b - b_2$. A little examination will show that, with the convention as to signs made in the statement of our proposition, these values hold, not only for the case exhibited in the figure, but whatever the positions of the points D, E, F. If, for instance, D lie in BC produced then BD is positive and m negative; but as $b + cm$ is then also negative (for the abscissa of D, $bc/(b + cm)$, is then negative), BD still has the value given above (rather than its negative).

The right side of our proposed equation is now equal to

$$\frac{\left[c_1\dfrac{acm}{b + cm}(b - b_2) - b_2\dfrac{ab}{b + cm}(c - c_1)\right]^2}{\left[ab - (b - b_2)\dfrac{ab}{b + cm}\right](bc - b_2c_1)\left[ca - (c - c_1)\dfrac{acm}{b + cm}\right]}.$$

As this reduces at once to the value found above for Δ''/Δ, our proposition is proved.

It may be noted that for the special case where the three lines drawn through the vertices meet in a point we get the familiar property $AF \cdot BD \cdot CE = AE \cdot CD \cdot BF$.

II. History of the Problem and Cesàro's Solution, by R. C. Archibald, Brown University.

The result of this problem, in its corrected form, was given by E. Cesàro in an article published more than twenty-five years ago.[1] It is, however, equivalent to the following, given twelve years earlier by C. Herkema[2]:

$$\frac{\Delta''}{\Delta} = \frac{n^2p^2(m - q)^2}{(mn + mp + np)(mp + np + nq)(np + nq + pq)},$$

where $BD/DC = n/p$, $CE/EA = p/q$, $AF/FB = m/n$. On the special assumption that

---

[1] "Etude de transversales," *Mathesis*, April, 1884, Volume 4, pp. 85–86.

[2] *Nouvelles annales de mathématiques*, 1872, (2) Volume 11, p. 477.

$AF = k \cdot AB$, $BD = k \cdot BC$, $CE = k \cdot CA$, T. Clausen showed in 1832[1] that

$$\frac{\Delta''}{\Delta} = \frac{1 - 4k + 4k^2}{1 - k + k^2}.$$

In the same number of "Crelle" (page 202) Steiner generalized this result.[2] He showed that if $AF = \alpha \cdot AB$, $BD = \beta \cdot BC$, $CE = \gamma \cdot CA$,

$$\frac{\Delta''}{\Delta} = 1 - \frac{(1-\alpha)\gamma}{1 - \alpha + \alpha\gamma} - \frac{(1-\beta)\alpha}{1 - \beta + \beta\alpha} - \frac{(1-\gamma)\beta}{1 - \gamma + \gamma\beta}.$$

In a problem proposed for solution Cesàro gave yet another form of expression[3] for the ratio of the areas: "Three lines drawn through the vertices of a triangle determine on the opposite sides six segment such that the difference between the product of three non-consecutive segments and the product of the three others is

$$\frac{abc}{a'b'c'}\left(\frac{\Delta''}{\Delta}\right)^2 lmn.$$

In this expression a', b', c' denote the sides of the triangle formed by the three lines; l, m, n the segments of these lines, contained between the vertices and the sides of the first triangle." [4]

The methods of solution of our problem by Steiner and Cesàro are those of elementary geometry. As Cesàro's solution is in a source inaccessible to many MONTHLY readers it is herewith reproduced:

Let $BD = a_1$, $CD = a_2$, $CE = b_1$, $AE = b_2$, $AF = c_1$, $BF = c_2$.

Let us seek first the ratios of the segments determined on each of the lines AD, BE, CF by the two others. The triangles ADC, ADB are cut respectively by the transversals BE, CF, giving

$$\frac{AC'}{C'D} = \frac{ab_2}{a_1b_1}, \qquad \frac{AB'}{B'D} = \frac{ac_1}{a_2c_2};$$

whence

$$\frac{AC'}{AD} = \frac{ab_2}{a_1b_1 + ab_2}, \qquad \frac{AB'}{AD} = \frac{ac_1}{a_2c_2 + ac_1}, \tag{1}$$

Now $a_1b_1 + ab_2 = ab - a_2b_1$, $a_2c_2 + ac_1 = ac - a_1c_2$, consequently,

$$\frac{AC'}{AD} = \frac{ab_2}{ab - a_2b_1}, \qquad \frac{AB'}{AD} = \frac{ac_1}{ca - c_2a_1}. \tag{2}$$

Subtracting one of the equalities (2) from the other

$$\frac{a'}{AD} = \frac{a(a_1b_1c_1 - a_2b_2c_2)}{(ab - a_2b_1)(ca - c_2a_1)}. \tag{3}$$

We have also

$$\frac{\Delta''}{AB'C} = \frac{A'B' \cdot C'B'}{A\,B' \cdot B'C}, \qquad \frac{AB'C}{ADC} = \frac{AB'}{AD}, \qquad \frac{ADC}{\Delta} = \frac{a_2}{a},$$

from which by multiplication,

$$\frac{\Delta''}{\Delta} = \frac{C'B'}{AD} \cdot \frac{A'B'}{B'C} \cdot \frac{a_2}{a}. \tag{4}$$

---

[1] Crelle's *Journal*, 1832, Volume 3, pp. 197–198.

[2] See also Steiner's *Gesammelte Werke*, Volume 1, 1881, p. 166; slips in the result as printed in Crelle's *Journal* are here corrected.

[3] *Nouvelle correspondance mathématique*, Volume 6, March, 1880, pp. 143–144, Question 545.

[4] Neuberg seemed to be the first to show (*Nouvelle correspondance mathématique*, Oct., 1880, Volume 6, p. 472) that

$$\frac{\Delta'}{\Delta} = \frac{a_1b_1c_1 + a_2b_2c_2}{(a_1 + a_2)(b_1 + b_2)(c_1 + c_2)},$$

where $AF/FB = c_1/c_2$, $BD/DC = a_1/a_2$, $CE/EA = b_1/b_2$, and Δ' is the area of the triangle DEF. This result was also stated by Gentry in *Nouvelles annales de mathématiques*, November, 1880, Vol. 39 (2), p. 528.

By means of the equality (3), and analogues to (2) and (3)

$$\frac{CB'}{CF} = \frac{ca_2}{ca - c_2a_1}, \qquad \frac{A'B'}{CF} = \frac{c(a_1b_1c_1 - a_2b_2c_2)}{(ca - c_2a_1)(bc - b_2c_1)},$$

we deduce from (4)

$$\frac{\Delta''}{\Delta} = \frac{(a_1b_1c_1 - a_2b_2c_2)^2}{(bc - b_2c_1)(ca - c_2a_1)(ab - a_2b_1)}.$$

340 (Calculus) [1913, 196; 1919, 213]. Proposed by C. N. SCHMALL, New York, N. Y.

A pencil of parallel rays of light is incident upon a lens whose faces have the radii r_1, r_2, respectively. Show that the distance of the principal focus from the center of the first face of the lens will be a maximum or a minimum when

$$\frac{r_1}{r_2} = \frac{(\mu - 1)^{1/2}}{1 + (\mu - 1)^{1/2}},$$

where μ has its usual meaning.

SOLUTION BY H. S. UHLER, Yale University.

It is necessary to remark at the very beginning that the given result appears incorrect as it does not involve the thickness of the lens, and the statement of the problem is ambiguous since it does not specify whether the first or the second principal focus is meant.

By straightforward analysis, involving Snell's law, but too long to deserve publication in this place, I found

$$x = t + \frac{r_2}{\mu - 1} - \frac{\mu r_2^2}{(\mu - 1)[\mu(r_1 + r_2) - (\mu - 1)t]}, \tag{1}$$

where, to fix the ideas, a double convex lens was contemplated, $x \equiv$ distance from the center of the first (or incidence) face to the second principal focus (the focus beyond the emergence face), $t \equiv$ axial thickness of lens, $\mu \equiv$ index of refraction of the material (glass) of the lens with respect to that of the surrounding medium (air), and r_1 and r_2 denote the arithmetical values of the radii of curvature of the first and second faces of the lens, respectively.

Formula (1) may be checked by referring to James P. C. Southall's *The Principles and Methods of Geometrical Optics*, either edition, page 275, equation (170); and by changing his d, n, and r_2 to t, μ, and $-r_2$, respectively.

Now what quantity is to be the independent variable? Let it be assumed that t is to vary while μ, r_1, and r_2 remain constant. Then the necessary condition for a maximum or a minimum is

$$\frac{dx}{dt} = 1 - \frac{\mu r_2^2}{[\mu(r_1 + r_2) - (\mu - 1)t]^2} = 0,$$

which leads directly to

$$t = \frac{\sqrt{\mu}[r_1\sqrt{\mu} + r_2(\sqrt{\mu} \pm 1)]}{\mu - 1}. \tag{2}$$

To determine which sign has physical meaning we may proceed as follows: $x - t$ must not be negative, otherwise the principal focus will fall inside the material of the lens. Substituting in (1) the expression for $(\mu - 1)\,t$ given by (2) we find

$$x - t = \frac{r_2}{\sqrt{\mu} - 1} \quad \text{or} \quad -\frac{r_2}{\sqrt{\mu} + 1},$$

according as the upper or lower sign in (2) be taken. Since, under all ordinary conditions, $\mu > 1$ the upper sign alone is acceptable.

Continuing,

$$\frac{d^2x}{dt^2} = -\frac{2\mu(\mu - 1)r_2^2}{[\mu(r_1 + r_2) - (\mu - 1)t]^3}$$

which, for the critical value given by (2), reduces to

$$\left(\frac{d^2x}{dt^2}\right)_2 = +\frac{2(\mu - 1)}{r_2\sqrt{\mu}}$$